Self-Assembly of Discrete Self-Similar Fractals (Extended Abstract)*

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Abstract

In this paper, we search for *absolute* limitations of the Tile Assembly Model (TAM), along with techniques to work around such limitations. Specifically, we investigate the self-assembly of fractal shapes in the TAM. We prove that no self-similar fractal fully weakly self-assembles at temperature 1, and that certain kinds of self-similar fractals do not strictly self-assemble at any temperature. Additionally, we extend the fiber construction from Lathrop et. al. (2007) to show that any self-similar fractal belonging to a particular class of "nice" self-similar fractals has a fibered version that strictly self-assembles in the TAM.

1 Introduction

Self-assembly is a bottom-up process by which (usually a small number of) fundamental components automatically coalesce to form a target structure. In 1998, Winfree [15] introduced the (abstract) Tile Assembly Model (TAM) - an extension of Wang tiling [13, 14], and a mathematical model of the DNA self-assembly pioneered by Seeman et. al. [11]. In the TAM, the fundamental components are un-rotatable, but translatable "tile types" whose sides are labeled with glue "colors" and "strengths." Two tiles that are placed next to each other *interact* if the glue colors on their abutting sides match, and they *bind* if the strength on their abutting sides matches, and is at least a certain "temperature." Rothemund and Winfree [10, 9] later refined the model, and Lathrop et. al. [7] gave a treatment of the TAM in which equal status is bestowed upon the self-assembly of infinite and finite structures. There are also several generalizations [2, 8, 5] of the TAM.

Despite its deliberate over-simplification, the TAM is a computationally and geometrically expressive model. For instance, Winfree [15] proved that the TAM is computationally universal, and thus can be directed algorithmically. Winfree [15] also exhibited a seven-tile-type self-assembly system, directed by a clever XOR-like algorithm, that "paints" a picture of a well-known shape, the discrete Sierpinski triangle **S**, onto the first quadrant. Note that the underlying *shapes* of each of the previous results are infinite canvases that cover the first quadrant, onto which computationally interesting shapes are painted (i.e., full weak self-assembly). Moreover, Lathrop et. al [6] recently gave a new characterization of the computably enumerable sets in terms of weak self-assembly using a "ray construction." It is natural to ask the question: How expressive is the TAM with respect to the self-assembly of a particular, possibly infinite shape, and nothing else (i.e., strict self-assembly)?

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In the case of strict self-assembly of finite shapes, the TAM certainly remains an interesting model, so long as the size (tile complexity) of the assembly system is required to be "small" relative to the shape that it ultimately produces. For instance, Rothemund and Winfree [10] proved that there are small tile sets in which large squares self-assemble. Moreover, Soloveichik and Winfree [12] established the remarkable fact that, if one is not concerned with the scale of an "algorithmically describable" finite shape, then there is always a small tile set in which the shape self-assembles. Note that if the tile complexity of an assembly system is unbounded, then every finite shape trivially (but perhaps not feasibly) self-assembles.

When the tile complexity of an assembly system is unbounded (yet finite), only infinite shapes are of interest. In the case of strict self-assembly of infinite shapes, the power of the TAM has only recently been investigated. Lathrop et. al. [7] established that self-similar tree shapes do not strictly self-assemble in the TAM given any finite number of tile types. A "fiber construction" is also given in [7], which strictly self-assembles a non-trivial fractal structure.

In this paper, we search for (1) *absolute* limitations of the TAM, with respect to the strict self-assembly of shapes, and (2) techniques that allow one to "work around" such limitations. Specifically, we investigate the strict self-assembly of fractal shapes in the TAM. We prove three main results: two negative and one positive. Our first negative (i.e., impossibility) result says that no self-similar fractal fully weakly self-assembles in the TAM at temperature 1. In our second impossibility result, we exhibit a class of discrete self-similar fractals, to which the standard discrete Sierpinski triangle belongs, that do not strictly self-assemble in the TAM (at *any* temperature). Finally, in our positive result, we use simple modified counters to extend the fiber construction from Lathrop et. al. [7] to a particular class of discrete self-similar fractals.

2 Preliminaries

2.1 The Tile Assembly Model

We work in the 2-dimensional discrete Euclidean space $\mathbb{Z}^2$. We write U_2 for the set of all *unit vectors*, i.e., vectors of length 1 in $\mathbb{Z}^2$. We write $[X]^2$ for the set of all 2-element subsets of a set X. All *graphs* here are undirected graphs, i.e., ordered pairs $G = (V, E)$, where V is the set of *vertices* and $E \subseteq [V]^2$ is the set of *edges*. A *grid graph* is a graph $G = (V, E)$ in which $V \subseteq \mathbb{Z}^2$ and every edge $\{\vec{a}, \vec{b}\} \in E$ has the property that $\vec{a} - \vec{b} \in U_2$. The *full grid graph* on a set $V \subseteq \mathbb{Z}^2$ is the graph $G^{\#}_V = (V, E)$ in which E contains *every* $\left\{\vec{a}, \vec{b}\right\} \in [V]^2$ such that $\vec{a} - \vec{b} \in U_2$.

We now give a brief sketch of the Tile Assembly Model. See [15, 10, 9, 7] for other developments of the model.

Intuitively, a tile type t is a unit square that can be translated, but not rotated, having a well-defined "side $\vec{u}$" for each $\vec{u} \in U_2$. Each side $\vec{u}$ of t has a "glue" of "color" $\mathrm{col}_t(\vec{u})$ - a string over some fixed alphabet Σ - and "strength" $\mathrm{str}_t(\vec{u})$ - a natural number - specified by its type t. Two tiles t and t' that are placed at the points $\vec{a}$ and $\vec{a} + \vec{u}$ respectively, *bind* with *strength* $\mathrm{str}_t(\vec{u})$ if and only if $(\mathrm{col}_t(\vec{u}), \mathrm{str}_t(\vec{u})) = (\mathrm{col}_{t'}(-\vec{u}), \mathrm{str}_{t'}(-\vec{u}))$.

Given a set T of tile types, an *assembly* is a partial function $\alpha : \mathbb{Z}^2 \dashrightarrow T$. An assembly is *stable* if it cannot be broken up into smaller assemblies without breaking bonds of total strength at least τ.

Self-assembly begins with a *seed assembly* σ and proceeds asynchronously and nondeterministically, with tiles adsorbing one at a time to the existing assembly in any manner that preserves stability at all times. A *tile assembly system* (*TAS*) is an ordered triple $\mathcal{T} = (T, \sigma, \tau)$, where T is a finite set of tile types, σ is a seed assembly with finite domain, and $\tau = 2$ is the temperature. An assembly α is *terminal*, and we write $\alpha \in \mathcal{A}_{\square}[\mathcal{T}]$, if no tile can be stably added to it. A TAS $\mathcal{T}$ is *directed*, or *produces a unique assembly*, if it has exactly one terminal assembly.

A set $X \subseteq \mathbb{Z}^2$ *weakly self-assembles* if there exists a TAS $\mathcal{T} = (T, \sigma, \tau)$ and a set $B \subseteq T$ such that $\alpha^{-1}(B) = X$ holds for every terminal assembly α. A set X *fully weakly self-assembles* if it weakly self-assembles in some TAS $\mathcal{T}$, and every terminal assembly of $\mathcal{T}$ tiles the entire plane. A set X *strictly self-assembles* if there is a TAS $\mathcal{T}$ for which every terminal assembly has domain X.

An *assembly sequence* in a TAS $\mathcal{T} = (T, \sigma, \tau)$ is an infinite sequence $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ of assemblies in which $\alpha_0 = \sigma$ and each α_{i+1} is obtained from α_i by the "τ-stable" addition of a single tile. To prove that a particular TAS $\mathcal{T} = (T, \sigma, \tau)$ is directed, it suffices to exhibit a *locally deterministic* [12] assembly sequence.

2.2 Discrete Self-Similar Fractals

In this subsection we introduce discrete self-similar fractals, and zeta-dimension [4].

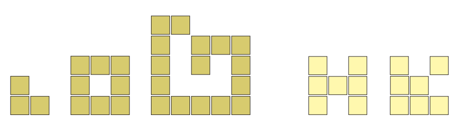
Definition. Let $1 < c \in \mathbb{N}$, and $X \subsetneq \mathbb{N}^2$ (we do not consider $\mathbb{N}^2$ to be a self-similar fractal). We say that X is a *c-discrete self-similar fractal*, if there is a set $\{(i,i) \mid i \in \{0, \ldots c-1\}\} \neq V \subseteq \{0, \ldots, c-1\} \times \{0, \ldots, c-1\}$ such that

$$X = \bigcup_{i=0}^{\infty} X_i,$$

where X_i is the i^{th} *stage* satisfying $X_0 = \{(0,0)\}$, and $X_{i+1} = X_i \cup \left(X_i + c^i V\right)$. In this case, we say that V *generates* X. X is a *discrete self-similar fractal* if it is a c-discrete self-similar fractal for some $c \in \mathbb{N}$.

In this paper, we are concerned with the following class of self-similar fractals.

Definition. A *nice discrete self-similar fractal* is a discrete self-similar fractal such that $(\{0, \ldots, c-1\} \times \{0\}) \cup (\{0\} \times \{0, \ldots, c-1\}) \subseteq V$, and $G^{\#}_{V}$ is connected.



(a) Nice (b) Non-nice

Figure 1: The first stages of discrete self-similar fractals. The fractals in (a) are nice, whereas (b) shows two non-nice fractals.

The most commonly used dimension for discrete fractals is zeta-dimension, which we use in this paper.

Definition. (Doty et. al. [4]) For each set $A \subseteq \mathbb{Z}^2$, the *zeta-dimension* of A is

$$\mathrm{Dim}_{\zeta}(A) = \limsup_{n \to \infty} \frac{\log |A_{\leq n}|}{\log n},$$

where $A_{\leq n} = \{(k,l) \in A \mid |k| + |l| \leq n\}$.

It is clear that $0 \leq \mathrm{Dim}_{\zeta}(A) \leq 2$ for all $A \subseteq \mathbb{Z}^2$.

3 Impossibility Results

In this section, we explore the theoretical limitations of the Tile Assembly Model with respect to the self-assembly of fractal shapes. First, we establish that no discrete self-similar fractal fully weakly self-assembles at temperature $\tau = 1$. Second, we exhibit a class $\mathcal{C}$ of discrete self-similar fractals, and prove that if $F \in \mathcal{C}$, then F does not strictly self-assemble in the TAM.

In this version of the paper, we merely state our results without proof. Full proofs of our results can be found at the following URL: `http://www.cs.iastate.edu/~lnsa`.

Definition. (Lathrop et. al. [7]) Let $G = (V, E)$ be a graph, and let $D \subseteq V$. For each $r \in V$, the *D-r-rooted subgraph* of G is the graph $G_{D,r} = (V_{D,r}, E_{D,r})$, where

$$V_{D,r} = \{v \in V \mid \text{every simple path from } v \text{ to (any vertex in) } D \text{ in } G \text{ goes through } r\}$$

and $E_{D,r} = E \cap [V_{D,r}]^2$. B is a *D-subgraph* of G if it is a D-r-rooted subgraph of G for some $r \in V$.

Definition. Let $G = (V, E)$ be a graph. Fix a set $D \subseteq V$, and let $r, r' \in V$.

1. (Adleman et. al. [1]) $G_{D,r}$ is *isomorphic* to $G_{D,r'}$, and we write $G_{D,r} \sim G_{D,r'}$ if there exists a vector $\vec{a} \in \mathbb{Z}^2$ such that $V_{D,r} = V_{D,r'} + \vec{a}$.

2. We say that $G_{D,r}$ is *unique* if $G_{D,r} \sim G_{D,r'} \Rightarrow r = r'$.

We will use the following technical result to prove that no self-similar fractal fully weakly self-assembles at temperature $\tau = 1$.

Lemma 1. (Adleman et. al. [1]) Let $X \subsetneq \mathbb{N}^2$ such that $G^{\#}_X$ is a finite tree, and assume that X strictly self-assembles in the TAS $\mathcal{T} = (T, \sigma, \tau)$. Let $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$. If $\alpha(\vec{u}) = \alpha(\vec{v})$, then the $G_{\text{dom } \sigma, \vec{u}} \sim G_{\text{dom } \sigma, \vec{v}}$.

We now have the machinery to prove our first impossibility result.

Theorem 1. If $F \subsetneq \mathbb{N}^2$ is a discrete self-similar fractal, and F fully weakly self-assembles in the TAS $\mathcal{T}_F = (T, \sigma, \tau)$, where σ consists of a single tile placed at the origin, then $\tau > 1$.

Note that Theorem 1 says that even if one is allowed to place a tile at *every* location in the first quadrant, it is still impossible for self-similar fractals to weakly self-assemble at temperature 1.

Next, we exhibit a class $\mathcal{C}$ of (non-tree) "pinch-point" discrete self-similar fractals that do not strictly self-assemble. Before we do so, we establish the following lower bound.

Lemma 2. If $X \subseteq \mathbb{Z}^2$ strictly self-assembles in the TAS $\mathcal{T} = (T, \sigma, \tau)$, where σ consists of a single tile placed at the origin, then $|T| \geq \left| \left\{ B \;\middle|\; B \text{ is a unique dom } \sigma\text{-subgraph of } G^{\#}_X \right\} \right|$.

Lemma 2 is not as tight as possible, but it applies to a general class of fractals. Our second impossibility result is the following.

Theorem 2. If $X \subsetneq \mathbb{N}^2$ is a discrete self-similar fractal satisfying (1) $\{(0,0), (0, c-1), (c-1, 0)\} \subseteq V$, (2) $V \cap (\{1, \ldots c-1\} \times \{c-1\}) = \emptyset$, (3) $V \cap (\{c-1\} \times \{1, \ldots, c-1\}) = \emptyset$, and (4) $G^{\#}_V$ is connected, then X does not strictly self-assemble in the Tile Assembly Model.

Corollary 1 (Lathrop, et. al. [7])**.** The standard discrete Sierpinski triangle $\mathbf{S}$ does not strictly self-assemble in the Tile Assembly Model.

4 Every Nice Self-Similar Fractal Has a Fibered Version

In this section, given a nice c-discrete self-similar fractal $X \subsetneq \mathbb{N}^2$ (generated by V), we define its fibered counterpart $\mathbf{X}$. Intuitively, $\mathbf{X}$ is nearly identical to X, but each successive stage of $\mathbf{X}$ is slightly thicker than the equivalent stage of X (see Figure 2 for an example). Our objective is to define sets $F_0, F_1, \ldots \subseteq \mathbb{Z}^2$, sets $T_0, T_1, \ldots \subseteq \mathbb{Z}^2$, and functions $l, f, t : \mathbb{N} \to \mathbb{N}$ with the following meanings.

1. T_i is the i^{th} stage of our construction of the fibered version of X.
2. F_i is the *fiber* associated with T_i. It is the smallest set whose union with T_i has a vertical left edge and a horizontal bottom edge, together with one additional layer added to these two now-straight edges.
3. $l(i)$ is the length (number of tiles in) the left (or bottom) edge of $T_i \cup F_i$.
4. $f(i) = |F_i|$.
5. $t(i) = |T_i|$.

These five entities are defined recursively by the equations

$$
\begin{aligned}
&T_0 = X_2 \text{ (the third stage of } X\text{)},\\
&F_0 = \left(\{-1\} \times \left\{-1, \ldots, c^2\right\}\right) \cup \left(\left\{-1, \ldots, c^2\right\} \times \{-1\}\right),\\
&l(0) = c^2 + 1,\ f(0) = 2c^2 + 1,\ t(0) = (|V| + 1)^2,\\
&T_{i+1} = T_i \cup \left((T_i \cup F_i) + l(i)V\right),\\
&F_{i+1} = F_i \cup \left(\{-i-2\} \times \{-i-2, -i-1, \cdots, l(i+1) - i - 3\}\right)\\
&\ \cup \left(\{-i-2, -i-1, \cdots, l(i+1) - i - 3\} \times \{-i-2\}\right),\\
&l(i+1) = c \cdot l(i) + 1,\\
&f(i+1) = f(i) + c \cdot l(i+1) - 1,\\
&t(i+1) = |V| t(i) + f(i).
\end{aligned}
$$

Finally, we let

$$\mathbf{X} = \bigcup_{i=0}^{\infty} T_i.$$

Note that the set $T_i \cup F_i$ is the union of an "outer framework," with an "internal structure." One can view the outer framework of $T_i \cup F_i$ as the union of a square S_i (of size $i+2$), a rectangle X_i (of height $i+2$ and width $l(i) - (i+2)$), and a rectangle Y_i (of width $i+2$ and height $l(i) - (i+2)$). Moreover, one can show that the internal structure of $T_i \cup F_i$ is simply the union of (appropriately-translated copies) of smaller and smaller X_i and Y_i-rectangles.

We have the following "similarity" between X and $\mathbf{X}$.

Lemma 3. If $X \subsetneq \mathbb{N}^2$ is a nice self-similar fractal, then $\mathrm{Dim}_\zeta(X) = \mathrm{Dim}_\zeta(\mathbf{X})$.

In the next section we sketch a proof that the fibered version of every nice self-similar fractal strictly self-assembles.

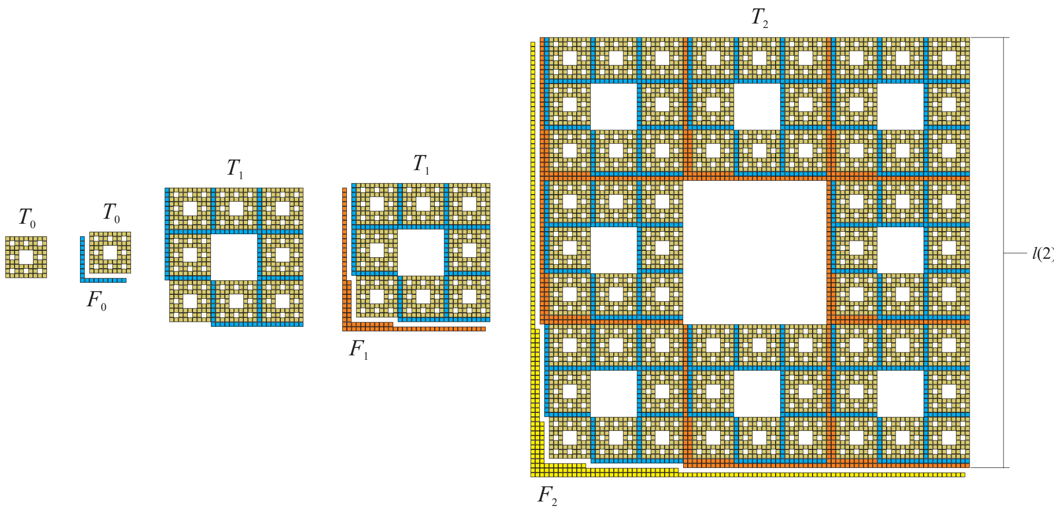


Figure 2: Construction of the fibered Sierpinski carpet. The blue, orange, and yellow tiles represent (possibly translated copies of) F_0, F_1, and F_2, respectively.

5 Sketch of Main Construction

Our second main theorem says that the fibered version of every nice self-similar fractal strictly self-assembles in the Tile Assembly Model (regardless of whether the latter strictly self-assembles).

Theorem 3. For every nice self-similar fractal $X \subset \mathbb{N}^2$, there is a directed TAS in which $\mathbf{X}$ strictly self-assembles.

We now give a brief sketch of our construction of the singly-seeded TAS $\mathcal{T}_{\mathbf{X}} = (X_{\mathbf{X}}, \sigma, 2)$ in which $\mathbf{X}$ strictly self-assembles. The full construction is implemented in C++, and is available at the following URL: `http://www.cs.iastate.edu/~lnsa`.

Throughout our discussion, $S_{\vec{u}}$, $Y_{\vec{u}}$, and $X_{\vec{u}}$ refer to the square, the vertical rectangle and the horizontal rectangle, respectively, that form the "outer framework" of the set $((T_i \cup F_i) + l(i) \cdot \vec{u})$ (See the right-most image in Figure 4).

5.1 Construction Phase 1

Here, directed graphs are considered. Let X be a nice (c-discrete) self-similar fractal generated by V. We first compute a directed spanning tree $B = (V, E)$ of $G_V^{\#}$ using a breadth-first search, and then compute the graph $B^{\mathrm{R}} = \left(V, E^{\mathrm{R}}\right)$, where

$$E^{\mathrm{R}} = \{(\vec{v}, \vec{u}) \mid (\vec{u}, \vec{v}) \in E \text{ and } \vec{u} \neq (0,0)\} \cup \{((0,1),(0,c-1)),((1,0),(c-1),0)\}.$$

Figure 3 depicts phase 1 of our construction for a particular nice self-similar fractal.

Notation. For all $\vec{0} \neq \vec{u} \in V$, $\vec{u}_{\text{in}}$ is the unique location $\vec{v}$ satisfying $(\vec{u}, \vec{v}) \in E^{\mathrm{R}}$.

5.2 Construction Phase 2

In the second phase we construct, for each $(0,0) \neq \vec{u} \in V$, a finite set of tile types $T_{\vec{u}}$ that self-assemble a particular subset of $\mathbf{X}$. There are two cases to consider.

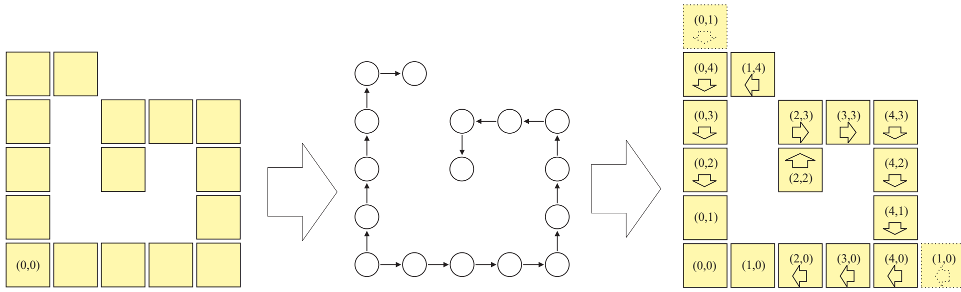


Figure 3: Phase 1 of our construction. Notice the two special cases (right-most image) in which we define $(0,1)_{\text{in}}$ and $(1,0)_{\text{in}}$.

Case 1 In the first case, we generate, for each $\vec{u} \in V - \{(0,0),(0,1),(1,0)\}$, three sets of tile types $T_{S_{\vec{u}}}$, $T_{X_{\vec{u}}}$, and $T_{Y_{\vec{u}}}$ that, when combined together, and assuming the presence of $((T_i \cup F_i) + l(i) \cdot \vec{u}_{\text{in}})$, self-assemble the set $((T_i \cup F_i) + l(i) \cdot \vec{u})$, for any $i \in \mathbb{N}$.

Case 2 In the second case, we generate, for each $\vec{u} \in \{(0,1),(1,0)\}$, the same three sets of tile types ($T_{S_{\vec{u}}}$, $T_{X_{\vec{u}}}$, and $T_{Y_{\vec{u}}}$) that self-assemble the set $((T_i \cup F_i) + l(i) \cdot \vec{u})$ "on top of" the set $((T_{i-1} \cup F_{i-1}) + l(i-1) \cdot \vec{u}_{\text{in}})$, for any $i \in \mathbb{N}$.

Finally, we let $T_{\mathbf{X}} = \bigcup_{(0,0) \neq \vec{u} \in V} T_{\vec{u}}$, where $T_{\vec{u}} = T_{S_{\vec{u}}} \cup T_{X_{\vec{u}}} \cup T_{Y_{\vec{u}}}$. Figure 4 gives a visual interpretation of the second phase of our construction. Our TAS is $\mathcal{T}_{\mathbf{X}} = (T_{\mathbf{X}}, \sigma, 2)$, where σ consists of a single "seed" tile type placed at the origin. Our full construction yields a tile set of 5983 tile types for the fractal generated by the points in the left-most image in Figure 4.

5.3 Details of Construction

Note that in our construction, the self-assembly of the sub-structures $S_{\vec{u}}$, $Y_{\vec{u}}$, and $X_{\vec{u}}$ can proceed either *forward* (away from the axes) or *backward* (toward the axes).

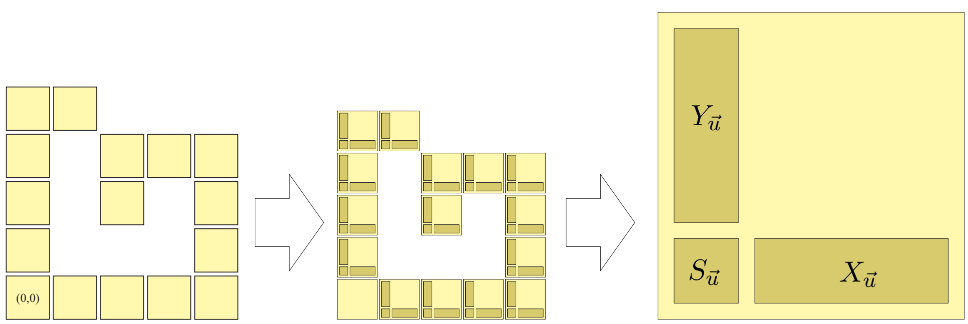


Figure 4: Let V be the left-most image. The first arrow represents phase 2 of the construction. The second arrow shows a magnified view of a particular point in V. Each point $(0,0) \neq \vec{u} \in V$ can be viewed conceptually as three components: the tile sets $T_{S_{\vec{u}}}$, $T_{X_{\vec{u}}}$ and $T_{Y_{\vec{u}}}$ that ultimately self-assemble the square $S_{\vec{u}}$, and the horizontal and vertical rectangles $X_{\vec{u}}$ and $Y_{\vec{u}}$ respectively.

5.3.1 Forward Growth

We now discuss the self-assembly of the set $((T_i \cup F_i) + \vec{u} \cdot l(i))$ for $\vec{u} \in V$ satisfying $\vec{u}_{\text{in}} \in (\vec{u} + \{(-1,0),(0,-1)\})$.

If $\vec{u} \notin \{(0,0),(0,1),(1,0)\}$ (i.e., case 1 of phase 2), then the tile set $T_{S_{\vec{u}}}$ self-assembles the square $S_{\vec{u}}$ directly on top (or to the right) of, and having the same width (height) as, the rectangle $Y_{\vec{u}_{\text{in}}}$ ($X_{\vec{u}_{\text{in}}}$). If $\vec{u} \in \{(0,1),(1,0)\}$ (i.e., case 2 of phase 2), then the tile set $T_{S_{\vec{u}}}$ self-assembles the square $S_{\vec{u}}$ on top (or to the right) of the set $Y_{\vec{u}_{\text{in}}}$ such that right (top) edge of the former is flush with that of the latter. Note that in case 2, the width of $Y_{\vec{u}_{\text{in}}}$ is always one less than that of $S_{\vec{u}}$. In either case, it is straightforward to construct such a tile set $T_{S_{\vec{u}}}$. The tile set of $T_{Y_{\vec{u}}}$ self-assembles a fixed-width base-c counter (based on the "optimal" binary counter presented in [3]) that, assuming a width of $i \in \mathbb{N}$, implements the following counting scheme: Count each positive integer j, satisfying $1 \leq j \leq c^i - 1$, in order but count each number exactly

$$[\![c \text{ divides } j]\!] \cdot \rho(j) + [\![c \text{ does not divide } j]\!] \cdot 1$$

times, where $\rho(j)$ is the largest number of consecutive least-significant 0's in the base-c representation of j, and $[\![\phi]\!]$ is the *Boolean* value of the statement ϕ. The *value* of a row is the number that it represents. We refer to any row whose value is a multiple of c as a *spacing row*. All other rows are *count* rows. The *type* of the counter that self-assembles $Y_{\vec{u}}$ is $\vec{u}$.

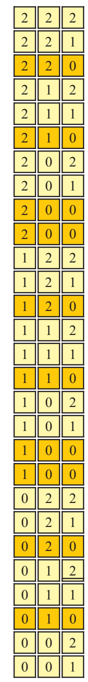


Figure 5: Example of a base-3 modified binary counter. The darker shaded rows are the spacing rows.

Each counter self-assembles on top (or to the right) of the square $S_{\vec{u}}$, with the width of the counter being determined by that of the square. It is easy to verify that if the width of $S_{\vec{u}}$ is $i+2$, then $T_{Y_{\vec{u}}}$ self-assembles a rectangle having a width of $i+2$ and a height of

$$\left(c^2+1\right)c^i + \frac{c^i - 1}{c-1} = l(i) - (i+2),$$

which is exactly $Y_{\vec{u}}$. Figure 5 shows the counting scheme of a base-3 counter of width 3. We construct the set $T_{X_{\vec{u}}}$ by simply reflecting the tile types in $T_{Y_{\vec{u}}}$ about the line $y = x$, whence the three sets of tile types $T_{S_{\vec{u}}}$, $T_{X_{\vec{u}}}$, and $T_{Y_{\vec{u}}}$ self-assemble the "outer framework" of the set $((T_i \cup F_i) + \vec{u} \cdot l(i))$.

The "internal structure" of the set $((T_i \cup F_i) + \vec{u} \cdot l(i))$ self-assembles as follows. Oppositely oriented counters attach to the right side of each contiguous group of spacing rows in the counter (of type $\vec{u}$) that self-assembles $Y_{\vec{u}}$. The number of such spacing rows determines the height of the horizontal counter, and its type is $(0, j/c \mod c)$, where j is the value of the spacing rows to which it attaches. We also hard code the glues along the right side of each non-spacing row to self-assemble the internal structure of the points in the set T_0.

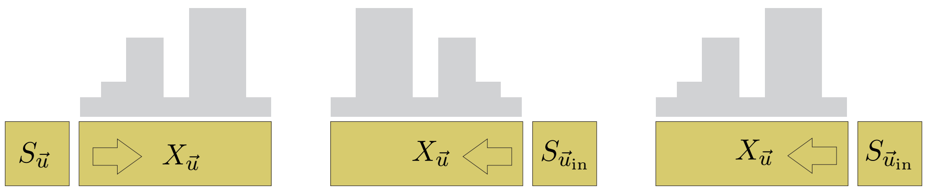
The situation for $X_{\vec{u}}$ is similar (i.e., a reflection of its vertical counterpart), with the exception that the glues along the top of each non-spacing row are configured differently than they were for $Y_{\vec{u}}$. This is because nice self-similar fractals need not be symmetric.

One can prove that, by recursively attaching smaller oppositely-oriented counters (of the appropriate type) to larger counters in the above manner, the internal structure of $((T_i \cup F_i) + \vec{u} \cdot l(i))$ self-assembles.

5.3.2 Reverse Growth

We now discuss the self-assembly of the set $((T_i \cup F_i) + \vec{u} \cdot l(i))$, for all $\vec{u} \in V$ satisfying $\vec{u}_{\text{in}} \in (\vec{u} + \{(1,0),(0,1)\})$.

In this case, the tile set $T_{Y_{\vec{u}}}$ ($T_{X_{\vec{u}}}$) self-assembles the set $Y_{\vec{u}}$ ($X_{\vec{u}}$) directly below (or to the left of) the square $S_{\vec{u}_{\text{in}}}$, and grows toward the x-axis (or y-axis) according to the base-c counting scheme outlined above. We also configure $T_{Y_{\vec{u}}}$ ($T_{X_{\vec{u}}}$) so that the right (or top)-most edge of $Y_{\vec{u}}$ ($X_{\vec{u}}$) is essentially the "mirror" image of its forward growing counterpart (See Figure 6). This last step ensures that the internal structure of $((T_i \cup F_i) + \vec{u} \cdot l(i))$ self-assembles correctly. Next, the square $S_{\vec{u}}$ attaches to the bottom (or left)-most edge of $Y_{\vec{u}}$ ($X_{\vec{u}}$). Finally, the set $X_{\vec{u}}$ ($Y_{\vec{u}}$) self-assembles via forward growth from the left (or top) of the square $S_{\vec{u}}$.



(a) (b) (c)

Figure 6: (a) depicts forward growth, (b) shows what happens if the tile set $T_{X_{\vec{u}}}$ were to simply "count in reverse," and (c) is the desired result.

5.3.3 Proof of Correctness

To prove the correctness of our construction, we use a local determinism argument. The details of the proof are technical, and therefore omitted from this version of the paper.

6 Conclusion

In this paper, we (1) established two new absolute limitations of the TAM, and (2) showed that fibered versions of "nice" self-similar fractals strictly self-assemble. Our impossibility results motivate the following question: Is there a discrete self-similar fractal $X \subsetneq \mathbb{N}^2$ that strictly self-assembles in the TAM? Moreover, our positive result leads us to ask: If $X \subsetneq \mathbb{N}^2$ is a discrete self-similar fractal, then is it always the case that X has a "fibered" version $\mathbf{X}$ that strictly self-assembles, and that is similar to X in some reasonable sense?

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