Self-Assembly of Infinite Structures: A Survey [2]

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Abstract

We survey some recent results related to the self-assembly of infinite structures in Winfree's abstract Tile Assembly Model. These results include impossibility results, as well as the construction of novel tile assembly systems that produce computationally interesting shapes and patterns. Several open questions are also presented and motivated.

1 Introduction

Self-assembly is a bottom-up process by which a small number of fundamental components automatically coalesce to form a target structure. In 1998, Winfree [33] introduced the (abstract) Tile Assembly Model (TAM) – an "effectivization" of Wang tiling [31,32] – as an over-simplified mathematical model of the DNA self-assembly pioneered by Seeman [27]. In the TAM, the fundamental components are un-rotatable, but translatable square "tile types" whose sides are labeled with glue "colors" and "strengths." Two tiles that are placed next to each other *interact* if the glue colors on their abutting sides match, and they *bind* if the strength on their abutting sides matches with total strength at least a certain ambient "temperature," usually taken to be 1 or 2.

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Rothemund and Winfree [24, 25] later refined the model, and Lathrop, Lutz, and Summers [18] gave a treatment of the TAM in which the self-assembly of infinite and finite structures can be unified under a single definition. See also [1, 23, 29]. There are also generalizations [8, 14, 20] of the abstract model.

Despite its deliberate over-simplification, the TAM is a computationally and geometrically expressive model at temperature 2. The reason is that, at temperature 2, certain tiles are not permitted to bind until *two* tiles are already present to match the glues on the bonding sides, which enables cooperation between different tile types to control the placement of new tiles. Winfree [33] proved that at temperature 2, the TAM is computationally universal and thus can be directed algorithmically.

Actual physical experimentation has driven lines of research involving kinetic variations of the TAM to deal with molecular concentrations, reaction rates, etc., as in [34], as well as work focused on error prevention and error correction [7, 28, 35]. For examples of the remarkable progress in the physical realization of self-assembling systems, see [26, 30].

Divergent from, but supplementary to, the laboratory work, much theoretical research involving the TAM has also been carried out. Interesting questions concerning the minimum number of tile types required to self-assemble shapes have been addressed in [3, 4, 25, 29]. Different notions of running time and bounds thereof were explored in [2, 5, 9]. Variations of the model where temperature values are intentionally fluctuated and the ensuing benefits and tradeoffs can be found in [4, 14]. Systems for generating randomized shapes or approximations of target shapes were investigated in [5, 10, 15]. This is just a small sampling of the theoretical work in the field of algorithmic self-assembly.

However, as different as they may be, the above mentioned lines of research share a common thread. They all tend to focus on the self-assembly of *finite* structures. Clearly, for experimental research this is a necessary limitation. Further, if the eventual goal of most of the theoretical research is to enable the development of fully functional, real world self-assembly systems, a valid question is: “Why care about anything other than finite structures?”

This paper surveys a collection of recent findings related to the self-assembly of *infinite* structures in the TAM. As a theoretical exploration of the TAM, this collection of results seeks to discover absolute limitations on the classes of shapes that self-assemble. These results also help to explore how fundamental aspects of the TAM, such as the inability of spatial locations to be reused and their immutability, affect and limit the constructions and computations that are achievable.

In addition to providing concise statements and intuitive descriptions of results, throughout this paper we define and motivate a set of open questions in the hope of furthering this line of research.

2 Preliminaries

2.1 The Tile Assembly Model

We work in the 2-dimensional discrete Euclidean space $\mathbb{Z}^2$. We write U_2 for the set of all *unit vectors*, i.e., vectors of length 1, in $\mathbb{Z}^2$. We regard the 4 elements of U_2 as (names of the cardinal) *directions* in $\mathbb{Z}^2$, namely (North, South, East, West).

We now give a brief and intuitive sketch of the Tile Assembly Model that is adequate for reading this paper. More formal details and discussion may be found in [18, 24, 25, 33]. Our notation is that of [18].

A *grid graph* is a graph $G = (V, E)$ in which $V \subseteq \mathbb{Z}^2$ and every edge $\{\vec{a}, \vec{b}\} \in E$ has the property that $\vec{a} - \vec{b} \in U_2$. The *full grid graph* on a set $V \subseteq \mathbb{Z}^2$ is the graph $G^{\#}_V = (V, E)$ in which E contains *every* $\{\vec{a}, \vec{b}\} \in [V]^2$ such that $\vec{a} - \vec{b} \in U_2$.

Intuitively, a tile type t is a unit square that can be translated, but not rotated, so it has a well-defined "side $\vec{u}$" for each $\vec{u} \in U_2$. Each side $\vec{u}$ is covered with a "glue" of "color" $\mathrm{col}_t(\vec{u})$ and "strength" $\mathrm{str}_t(\vec{u})$ specified by its type t. Tiles are depicted as squares whose various sides have zero, one or two notches, indicating whether the glue strengths on these sides are 0, 1, or 2, respectively. If two tiles are placed with their centers at adjacent points $\vec{m}, \vec{m} + \vec{u} \in \mathbb{Z}^2$, where $\vec{u} \in U_2$, and if their abutting sides have glues that match in both color and strength, then they form a *bond* with this common strength. If the glues do not so match, then no bond is formed between these tiles. In this paper, all glues have strength 0, 1, or 2. Each side's "color" is indicated by an alphanumeric label. Given a set T of tile types and a "temperature" $\tau \in \mathbb{N}$, a τ-T-*assembly* is a partial function $\alpha : \mathbb{Z}^2 \dashrightarrow T$ (intuitively, a placement of tiles at some locations in $\mathbb{Z}^2$) that is τ-*stable* in the sense that it cannot be "broken" into smaller assemblies without breaking bonds of total strength at least τ. If α and α' are assemblies, then α is a *subassembly* of α', and we write $\alpha \sqsubseteq \alpha'$, if $\mathrm{dom}\ \alpha \subseteq \mathrm{dom}\ \alpha'$ and $\alpha(\vec{m}) = \alpha'(\vec{m})$ for all $\vec{m} \in \mathrm{dom}\ \alpha$.

Self-assembly begins with a *seed assembly* σ and proceeds asynchronously and nondeterministically, with tiles adsorbing one at a time to the existing assembly in any manner that preserves τ-stability at all times. A *tile assembly system* (*TAS*) is an ordered triple $\mathcal{T} = (T, \sigma, \tau)$, where T is a finite set of tile types, σ is a seed assembly with finite domain, and $\tau \in \mathbb{N}$. A *generalized* *tile assembly system* (*GTAS*) is defined similarly, but without the finiteness requirements. We write $\mathcal{A}[\mathcal{T}]$ for the set of all assemblies that can arise (in finitely many steps or in the limit) from $\mathcal{T}$. An assembly α is *terminal*, and we write $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$, if no tile can be τ-stably added to it. An assembly sequence in a TAS $\mathcal{T}$ is a (finite or infinite) sequence $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots)$ of assemblies in which each α_{i+1} is obtained from α_i by the addition of a single tile. The *result* $\mathrm{res}(\vec{\alpha})$ of such an assembly sequence is its unique limiting assembly. (This is the last assembly in the sequence if the sequence is finite). The set $\mathcal{A}[\mathcal{T}]$ is partially ordered by the relation $\longrightarrow$ defined by

$$\alpha \longrightarrow \alpha' \Leftrightarrow \begin{array}{l}\text{there is an assembly}\\ \text{sequence } \vec{\alpha} = (\alpha_0, \alpha_1, \ldots)\\ \text{such that } \alpha_0 = \alpha \text{ and}\\ \alpha' = \mathrm{res}(\vec{\alpha}).\end{array}$$

We say that $\mathcal{T}$ is *directed* if the relation $\longrightarrow$ is directed, i.e., if for all $\alpha, \alpha' \in \mathcal{A}[\mathcal{T}]$, there exists $\alpha'' \in \mathcal{A}[\mathcal{T}]$ such that $\alpha \longrightarrow \alpha''$ and $\alpha' \longrightarrow \alpha''$. It is easy to show that $\mathcal{T}$ is directed if and only if there is a unique terminal assembly $\alpha \in \mathcal{A}[\mathcal{T}]$ such that $\sigma \longrightarrow \alpha$.

In general, even a directed TAS may have a very large (perhaps uncountably infinite) number of different assembly sequences leading to its terminal assembly. This seems to make it very difficult to prove that a TAS is directed. Fortunately, Soloveichik and Winfree [29] have recently defined a property, *local determinism*, of assembly sequences and proven the remarkable fact that, if a TAS $\mathcal{T}$ has *any* assembly sequence that is locally deterministic, then $\mathcal{T}$ is directed. Intuitively, a tile assembly system $\mathcal{T}$ is locally deterministic if (1) each tile added in $\mathcal{T}$ "just barely" binds to the existing assembly (meaning that when a tile binds, it does so by forming bonds whose strengths sum to *exactly* τ); and (2) if a tile of type t_0 at a location $\vec{m}$ and its immediate "OUT-neighbors" are deleted from some producible assembly of $\mathcal{T}$, then no tile of type $t \neq t_0$ can attach itself to the thus-obtained configuration at location $\vec{m}$ (effectively, tiles of only one type can bind in each location during assembly).

A set $X \subseteq \mathbb{Z}^2$ *weakly self-assembles* if there exists a TAS $\mathcal{T} = (T, \sigma, \tau)$ and a set $B \subseteq T$ (B constitutes the "black" tiles) such that $\alpha^{-1}(B) = X$ holds for every assembly $\alpha \in \mathcal{A}_\Box[\mathcal{T}]$.

Open Problem 2.1 *If $X \subseteq \mathbb{Z}^2$ weakly self-assembles in a directed tile assembly system, does X weakly self-assemble in a locally deterministic tile assembly system?*

A set X *strictly self-assembles* if there is a TAS $\mathcal{T}$ for which every assembly $\alpha \in \mathcal{A}_{\Box}[\mathcal{T}]$ satisfies dom $\alpha = X$. Note that if X strictly self-assembles, then X weakly self-assembles. (Let all tiles be black.)

Open Problem 2.2 *If $X \subseteq \mathbb{Z}^2$ weakly self-assembles, does X weakly self-assemble in a directed TAS $\mathcal{T}$?*

The previous open problem seeks to determine the "power of nondeterminism" in the abstract Tile Assembly Model with respect to the weak self-assembly of infinite patterns. It is worthy of note that Open Problem 2.2–with "weakly" replaced by "strictly"–was recently solved by Nathaniel Bryans, Ehsan Chiniforooshan, David Doty, Lila Kari, and Shinnosuke Seki [6], who showed that there are shapes which strictly self-assemble but which can *only* do so in undirected TAS's. The interested reader is highly encouraged to consult the aforementioned reference for further open problems related to the power of nondeterminism in self-assembly.

2.2 Discrete Self-Similar Fractals

In this subsection we introduce discrete self-similar fractals and zeta-dimension.

Definition Let $1 < c \in \mathbb{N}$, and $X \subsetneq \mathbb{N}^2$. We say that X is a *c-discrete self-similar fractal*, if there is a (non-trivial) set $V \subseteq \{0, \ldots, c-1\} \times \{0, \ldots, c-1\}$ such that $X = \bigcup_{i=0}^{\infty} X_i$, where X_i is the i^{th} *stage* satisfying $X_0 = \{(0,0)\}$, and $X_{i+1} = X_i \cup (X_i + c^i V)$. In this case, we say that V *generates* X.

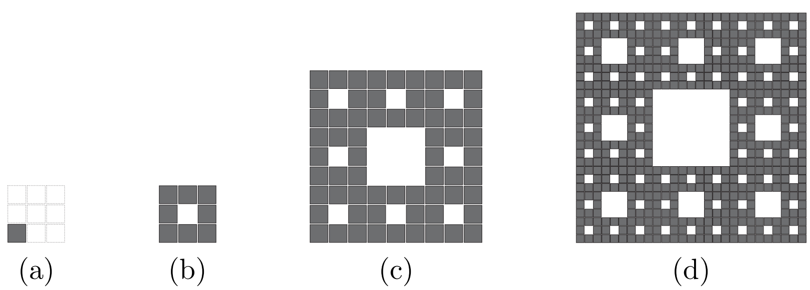


Fig. 1. The first four stages of the discrete "Sierpinski carpet" (X_0, $X_1 = V$, X_2, and X_3 are shown in (a), (b), (c), and (d) respectively). Note that (d) is scaled down.

The most commonly used dimension for discrete fractals is zeta-dimension, which we refer to in this paper. See [11] for a complete discussion of zeta-dimension.

Definition For each set $A \subseteq \mathbb{Z}^2$, the *zeta-dimension* of A is

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log |A_{\leq n}|}{\log n},$$

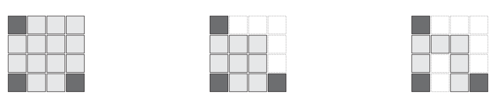
where $A_{\leq n} = \{(k,l) \in A \mid |k| + |l| \leq n\}$. Note that the set $A_{\leq n}$ is essentially the set of points one can reach by starting at the origin and taking at most n steps (a step being a movement in the upward or rightward direction). It is clear that $0 \leq \mathrm{Dim}_\zeta(A) \leq 2$ for all $A \subseteq \mathbb{Z}^2$.

3 Strict Self-Assembly

In searching for absolute limitations of the TAM with respect to the strict self-assembly of shapes, it is necessary to consider infinite shapes because any finite, connected shape strictly self-assembles via an inefficient spanning tree construction in which there is a unique tile type created for each point in the target shape. In this section we discuss (both positive and negative) results pertaining to the strict self-assembly of infinite shapes in the TAM.

3.1 The Impossibility of the Strict Self-Assembly of Pinch-point Discrete Self-Similar Fractals

In [22], Patitz and Summers defined a class $\mathcal{C}$ of (possibly well-connected non-tree) "pinch-point" discrete self-similar fractals, and proved that if $X \in \mathcal{C}$, then X does not strictly self-assemble in any directed tile assembly system at any temperature. The generator for a pinch-point fractal has exactly one point in each of its top-most and right-most rows, $(0, c)$ and $(c, 0)$, respectively. The other constraint is that the points in the generator are connected. See Figure 2 for an example.



(a) (b) (c)

Fig. 2. "Construction" of a pinch-point fractal generator; the dark gray points in (a) must be included; the white points in (b) in the top row and right column cannot be included; the generator must be connected.

A famous example of a pinch-point fractal is the standard discrete Sierpinski triangle **S**. The impossibility of the strict self-assembly of **S** was first shown in [18].

Theorem 3.1 *If $X \subsetneq \mathbb{N}^2$ is a pinch-point discrete self-similar fractal, then X does not strictly self-assemble in the Tile Assembly Model.*

The proof idea of Theorem 3.1 is simple: If such a pinch-point fractal were to strictly self-assemble in a finite tile system $\mathcal{T}$, then one could construct an infinite series of tile assembly systems $\mathcal{T}_0, \mathcal{T}_1, \ldots$ (from the tile types of $\mathcal{T}$) in which larger and larger finite shapes strictly self-assemble, contradicting the "finiteness" of $\mathcal{T}$. Theorem 3.1 begs the following question.

Open Problem 3.2 *Does any non-trivial discrete self-similar fractal strictly self-assemble in the TAM?*

We conjecture that the answer to the previous question is "no". However, proving that there exists a (non-trivial) discrete self-similar fractal that does strictly self-assemble would likely involve a novel and useful algorithm for directing the behavior of self-assembly. It is worthy of note that Patitz and Summers proved that no discrete self-similar fractal strictly self-assembles at temperature 1 in a locally deterministic TAS [22]

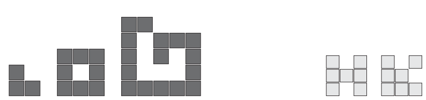
3.2 Strict Self-Assembly of Nice Discrete Self-Similar Fractals

As shown above, there is a class of discrete self-similar fractals that do not strictly self-assemble (at any temperature) in the TAM. However, in [22], Patitz and Summers introduced a particular set of "nice" discrete self-similar fractals that contains some but not all pinch-point discrete self-similar fractals. Further, they proved that any element of the former class has a "fibered" version that strictly self-assembles.

3.2.1 Nice Discrete Self-Similar Fractals

We now review the definition of a "nice" discrete self-similar fractal.

Definition A *nice discrete self-similar fractal* is a discrete self-similar fractal (generated by V) such that $(\{0,\ldots,c-1\}\times\{0\})\cup(\{0\}\times\{0,\ldots,c-1\}) \subseteq V$, and $G_V^{\#}$ is connected.



(a) Nice

(b) Non-nice

Fig. 3. The first stages of discrete self-similar fractals. The fractals in (a) are nice, whereas (b) shows two non-nice fractals.

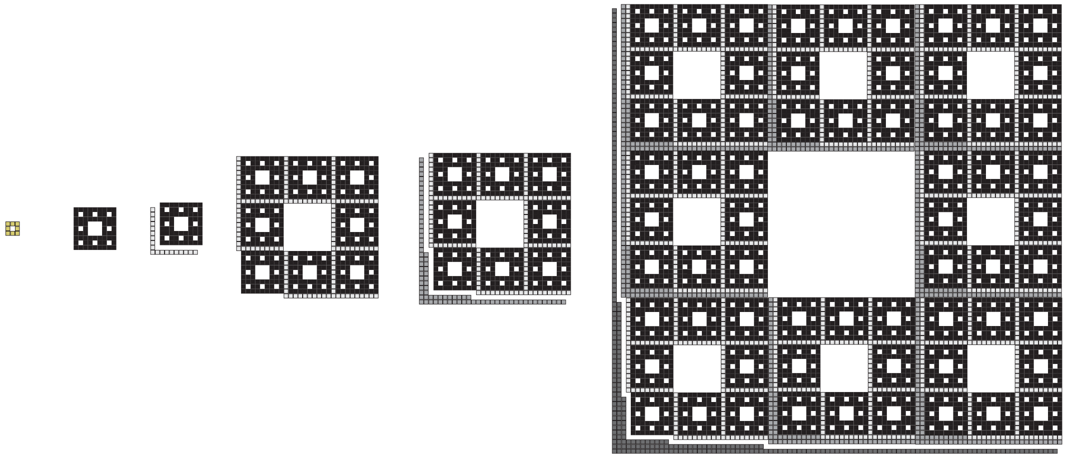


Fig. 4. Construction of the fibered Sierpinski carpet

3.2.2 Nice Fractals Have "Fibered" Versions

The inability of pinch-point fractals (and the conjectured inability of any discrete self-similar fractal) to strictly self-assemble in the TAM is based on the intuition that the necessary amount of information cannot be transmitted through available connecting tiles during self-assembly.

Thus, for any nice discrete self-similar fractal X, Patitz and Summers [22] defined a fiber operator $\mathcal{F}(X)$ (an extension of [18]) which adds, in a zeta-dimension-preserving manner, an asymptotically negligible amount of additional bandwidth to X. Intuitively, $\mathcal{F}(X)$ is nearly identical to X, but each successive stage of $\mathcal{F}(X)$ is slightly thicker than the equivalent stage of X. Figure 4 shows an example of the recursive construction of $\mathcal{F}(X)$, where X is the discrete Sierpinski carpet.

The following lemma testifies to the zeta-dimension preserving nature of $\mathcal{F}$.

Lemma 3.3 *If X is a nice self-similar fractal, then $Dim_\zeta(X) = Dim_\zeta(\mathcal{F}(X))$.*

The main positive result of [22] says that the fibered version of every nice self-similar fractal strictly self-assembles.

Theorem 3.4 *For every nice discrete self-similar fractal $X \subsetneq \mathbb{N}^2$, there exists a directed TAS in which $\mathcal{F}(X)$ strictly self-assembles.*

Strict self-assembly of $\mathcal{F}(X)$ is achieved via a "modified base-c counter" algorithm that is embedded into the aforementioned additional bandwidth of $\mathcal{F}(X)$. Moreover, the self-similar nature of counting results in the self-similar nature of $\mathcal{F}(X)$. At the time of this writing, it appears non-trivial to extend the fiber operator $\mathcal{F}$ to other discrete self-similar fractals such as the 'H' fractal (the second-to-the-right most image in Figure 3).

Additionally, in [19], Lutz and Shutters presented another zeta-dimension preserving construction which self-assembles an approximation of the Sierpinski triangle. However, their *laced Sierpinski triangle* is a superset of the Sierpinski triangle and thus forms each stage "in place" while building the necessary fibering inside of those stages.

Open Problem 3.5 *Does there exist a "fiber construction" $\mathcal{F}$ such that, for every discrete self-similar fractal X whose generator is connected, X and $\mathcal{F}(X)$ share the same zeta-dimension (or perhaps a stronger notion of mathematical similarity) and $\mathcal{F}(X)$ strictly self-assembles?*

4 Weak Self-Assembly

It is our contention that weak self-assembly captures the intuitive notion of what it means to "compute" with a tile assembly system. For example, the use of tile assembly systems to build shapes is captured by requiring all tile types to be black, in order to ask what set of integer lattice points contain any tile at all (so-called *strict self-assembly*). However, weak self-assembly is a more general notion. This section is devoted to the weak self-assembly of computationally and geometrically interesting sets.

4.1 Non-cooperative Self-Assembly

Temperature 1 tile assembly systems are desirable because, in current laboratory implementations of algorithmic self-assembly, strength 2 bonds are difficult to create. With that said, what kind of structures can temperature 1 tile assembly systems produce? In this section, we review a partial answer to this question.

4.1.1 Universal Computation at Temperature 1?

In [12], Doty, Patitz, and Summers establish that only the most "boring" of sets of integer lattice points weakly self-assemble in non-cooperative self-assembly systems, given a natural assumption. The formal definition of "boring" is as follows.

Definition A set $X \subseteq \mathbb{Z}^2$ is *semi-doubly periodic* if there exist three vectors $\vec{b}, \vec{u}, \vec{v} \in \mathbb{Z}^2$ such that $X = \left\{ \vec{b} + n\vec{u} + m\vec{v} \;\middle|\; n, m \in \mathbb{N} \right\}$.

The following observation justifies the intuition that finite unions of semi-doubly periodic sets constitute only the computationally simplest subsets of

$\mathbb{Z}^2$.

Observation 4.1 *Let $A \subseteq \mathbb{Z}^2$ be a finite union of semi-doubly periodic sets. Then the unary languages $L_{A,x} = \left\{ 0^{|x|} \;\middle|\; (x, y) \in A \text{ for some } y \in \mathbb{Z} \right\}$ and $L_{A,y} = \left\{ 0^{|y|} \;\middle|\; (x, y) \in A \text{ for some } x \in \mathbb{Z} \right\}$ consisting of the unary representations of the projections of A onto the x-axis and y-axis, respectively, are regular languages.*

So much for the hope of universal computation in non-cooperative self-assembly systems!

However, in order to prove that universal computation is impossible without cooperation, Doty, Patitz and Summers require the hypothesis that the tile system in question is *pumpable*. Informally, this means that every sufficiently long path of tiles in any assembly of this system contains a segment in which the same tile type repeats (a condition clearly implied by the pigeonhole principle), and that furthermore, there exists an assembly sequence in which the sub-path of tiles between these two occurrences can be repeated indefinitely ("pumped") along the same direction as the first occurrence of the segment, without "colliding" with a previous portion of the path. The main result of [13] is stated as follows.

Theorem 4.2 *Let $\mathcal{T} = (T, \sigma, 1)$ be a TAS that is directed and pumpable. If a set $X \subseteq \mathbb{Z}^2$ weakly self-assembles in $\mathcal{T}$, then X is a finite union of semi-doubly periodic sets.*

Open Problem 4.3 *Is every directed, temperature 1 tile assembly system that produces a two-dimensional infinite assembly pumpable?*

The implication of this open problem is that, if the answer is yes (as conjectured), then universal computation is impossible at temperature 1 in directed, 2-dimensional tile assembly systems. However, note that several unpublished constructions by other authors have demonstrated that universal computation *is* in fact possible by relaxing these constraints, either by allowing the use of the third dimension (in fact, only one additional plane) or probabilistic (non-directed) assembly.

4.2 Numerically Self-Similar Fractals

In [16], Kautz and Lathrop provide a uniform procedure for generating tile assembly systems in which discrete *numerically self-similar* fractals weakly self-assemble. This particular class of discrete self-similar fractals is defined in terms of the residues modulo a prime p of the entries in a two-dimensional matrix obtained from a simple recursive equation. Examples of numerically self-similar fractals are the Sierpinski triangle and the Sierpinski carpet.

Open Problem 4.4 *Does every discrete self-similar fractal weakly self-assemble?*

4.3 Decidable Sets

We now shift gears and discuss the weak self-assembly of 2-dimensional representations of (computable) sets of natural numbers at temperature 2.

4.3.1 A Characterization of Decidable Sets of Natural Numbers

Here, the story begins with [21], where Patitz and Summers exhibited a novel characterization of decidable sets of positive integers in terms of weak self-assembly, i.e., they proved the following theorem.

Theorem 4.5 *Let $A \subseteq \mathbb{N}$. The set A is decidable if and only if $\{0\} \times -A$ and $\{0\} \times (-A)^c$ weakly self-assemble.*

Theorem 4.5 is essentially the "self-assembly version" of the classical theorem which says that a set $A \subseteq \mathbb{N}$ is decidable if and only if A and A^c are computably enumerable Patitz and Summers [21] further exploit the underlying geometry of self-assembly and prove a "geometrically stronger" version of Theorem 4.5 as follows.

Theorem 4.6 *Let $A \subseteq \mathbb{N}$. The set A is decidable if and only if $\{0\} \times -A$ and $\{0\} \times (-A)^c$ weakly self-assemble and every tile is placed in the* first quadrant*.*

Patitz and Summers [21] also show that Theorem 4.6 holds for tile assembly systems that only place tiles in arbitrarily thin "pie slices" of the first quadrant with a corresponding blowup in tile complexity.

4.3.2 Some Decidable Sets Do Not Weakly Self-Assemble

In contrast to Theorem 4.5, Lathrop, Lutz, Patitz, and Summers [17] proved that there are decidable sets $D \subseteq \mathbb{Z}^2$ that do not weakly self-assemble.

Theorem 4.7 *There is a decidable set $D \subseteq \mathbb{Z}^2$ that does not weakly self-assemble where $D \in DTIME\left(2^{linear}\right)$.*

Is it possible to do any better?

Open Problem 4.8 *Is there a polynomial-time decidable set $D \in \mathbb{Z}^2$ such that D does not weakly self-assemble?*

Dovetailing is easy to do on a Turing machine because it is possible to reuse space. But how can one carry out "dovetailing" of computations in the tile assembly model where both space *and* time are "non-reusable" resources? A self-assembly version of dovetailing was developed and used by Lathrop, Lutz, Patitz and Summers in [17] to explore the impact of geometry on computability and complexity in self-assembly [17]. In particular, Lathrop, Lutz, Patitz and Summers proved that for every TM M, there exists a directed TAS that simulates M on every input $x \in \mathbb{N}$ in the two dimensional discrete Euclidean plane.

Theorem 4.9 *If $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a function such that $f(n) = \binom{n+1}{2} + (n+1)\lfloor \log n \rfloor + 6n - 2^{1+\lfloor \log(n) \rfloor} + 2$, then for all $A \subseteq \mathbb{Z}^+$, A is computably enumerable if and only if the set $X_A = \{(f(n), 0) \mid n \in A\}$ self-assembles.*

Intuitively, $\mathcal{T}_M$ self-assembles a "gradually thickening bar" immediately below the positive x-axis with upward growths emanating from well-defined intervals of points. For each $x \in \mathbb{Z}^+$, there is an upward growth, in which a modified wedge construction carries out a simulation of M on x. If M halts on x, then (a portion of) the upward growth associated with the simulation of $M(x)$ eventually stops, and sends a signal down along the right side of the upward growth via a one-tile-wide-path of tiles to the point $(f(x), 0)$, where a black tile is placed. In order to allow for an infinite number of simultaneous computations to occur, any of which may never halt and require infinite time and tape space, all without colliding with each other and leaving space for the "answers" to be correctly deposited at the locations specified by $(f(x), 0)$, intricate geometric techniques were required.

Open Problem 4.10 *Does Theorem 4.9 hold for some f where $f(n) = \Theta(n)$?*

We conjecture that the answer to the previous open problem is "no", and that the construction of [17] is effectively optimal.

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